Bifurcations of an age-structured predator-prey model

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Abstract

The relationships between a predator and its prey are subject of numerous studies in ecology. Since the first mathematical model, describing over time such trophic interactions was introduced ([3], [5]), the so-called Lotka Volterra equations, and predator-prey models are still a wide subject of study in population dynamics. For a better modelling, it is interesting to consider an age-structured prey population, leading to a PDE of transport type. Once the problem is well-posed, we use some spectral analysis results [2] and a stability analysis of the equilibria [6] to study the time asymptotic behaviour of the solutions. We prove in [4] the existence of two thresholds for the extinction and the explosion of both populations. Finally, using numerical computations, we show that other behaviours can appear.

1. Lotka-Volterra model

Let $(x(t), y(t))$ be the populations at time t of prey and predator. In the 1920s, Alfred Lotka [3] and Vito Volterra [5] proposed the ODE model

$$\begin{cases} x'(t) = ax(t) - bx(t)y(t), \\ y'(t) = cx(t)y(t) - dy(t), \end{cases}$$

with positive parameters and

- a is the **birth** rate of the prey:
- d is the **mortality** rate of the predator;
- the nonlinear term models the **predation**.

This models gives rise to periodic solution which explains the oscillation of lynx and snowshoe hare in Canada.

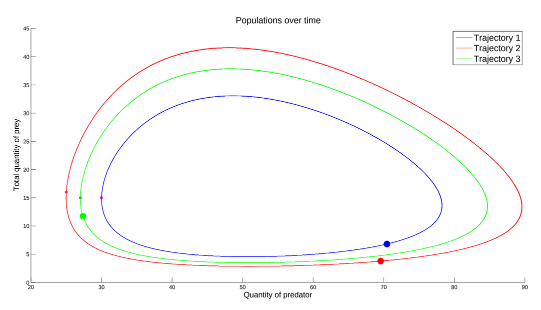

Figure 1: *Periodic solutions*

2. Age-structured model

Consider that birth and death of each prey depend on his **age** $(a \geq 0)$. We get the generalized PDE structured model

$$\begin{cases} \partial_t x + \partial_a x = -\mu(a)x - y(t)\gamma(a)x, \\ y'(t) = \alpha y(t) \int_0^\infty \gamma(a)x(t,a)da - \delta y(t), \\ x(t,0) = \int_0^\infty \beta(a)x(t,a)da, \\ x(0,.) = x_0, \quad y(0) = y_0, \end{cases} \quad (1)$$

with $\delta > 0$ and where

- $\mu, \gamma, \beta \in L^\infty_+(\mathbb{R}_+)$ are age-dependent functions;
- $\alpha \in (0,1)$ is the assimilation coefficient of ingested preys.

3. Well-posedness

We assume

$$\exists \mu_0 > 0 : \mu(a) \geq \mu_0 \text{ f.a.e. } a \geq 0. \quad \text{(H1)}$$

Let $X = L^1(\mathbb{R}_+) \times \mathbb{R}$ and X_+ his nonnegative cone. Consider the operator $A : D(A) \subset X \to X$, by

$$A\begin{pmatrix}\phi \\ z\end{pmatrix} = \begin{pmatrix}-\phi' - \mu\phi \\ -\delta z\end{pmatrix},$$

$$D(A) = \{(\phi, z) \in W^{1,1}(\mathbb{R}_+) \times \mathbb{R}, \phi(0) = \int_0^\infty \beta(a)\phi(a)da\}.$$

and define the function $f : X \to X$ by

$$f(\phi, z) = \begin{pmatrix}-z\gamma(.)\phi(.) \\ \alpha z \int_0^\infty \gamma(a)\phi(a)da\end{pmatrix}.$$

We then get the abstract Cauchy Problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} = A\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} + f(x(t), y(t)), \\ (x(0), y(0)) = (x_0, y_0) \in X. \end{cases}$$

Classical results of the semigroup theory proves that

Theorem 1. (Existence/uniqueness)
For all $(x_0, y_0) \in X_+$, Problem (1) has a unique mild solution $(x, y) \in \mathcal{C}(\mathbb{R}_+, X_+)$.

4. Equilibria

Define

$$a_1 = \sup\{a \geq 0 : |\text{supp}(\gamma) \cap (0,a)| = 0\} < \infty;$$

$$R_0 = \int_0^\infty \beta(a)e^{-\int_0^a \mu(s)ds}ds \text{ (extinction threshold)};$$

$$R_- = \int_0^{a_1} \beta(a)e^{-\int_0^a \mu(s)ds}da \text{ (unboundedness threshold)}.$$

Theorem 2. (Number of equilibrium)

$R_0 < 1$	$R_0 > 1, R_- < 1$	$R_- > 1$
E_0	E_0 and E_2	E_0

where $E_0 = (0,0)$ and E_2 is a positive steady state.

5. Stability analysis

Using classical results of spectral analysis ([2], [6]), we get the following characteristic equation for E_0

$$\int_0^\infty \beta(a)e^{-\int_0^a [\lambda + \mu(s)]ds}da = 1$$

which implies

Theorem 3. (Stability of E_0)
1. $R_0 < 1 \Rightarrow E_0$ is globally stable (Figure 2)
2. $R_0 > 1 \Rightarrow E_0$ is unstable.

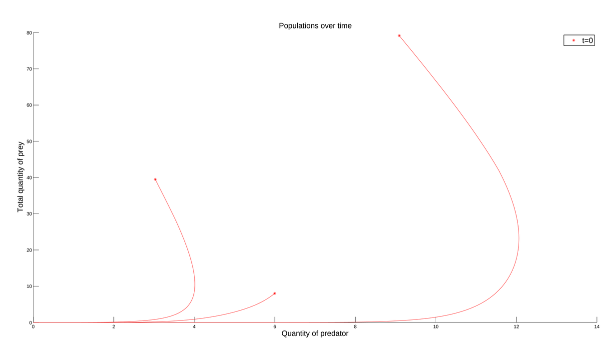

Figure 2: *Convergence to E_0 when $R_0 < 1$*

6. And for R0>1 ?

We suppose

$$\exists \eta_1 > 0, \exists\, 0 < \underline{a} < \overline{a} : \beta(a) \geq \eta_1, a \in (\underline{a}, \overline{a}) \quad \text{(H2)}$$

It means that preys of a certain range of age have a high ability to reproduce.
We find a bassin of attraction X_p such that

Theorem 4. (Unbounded solutions)
If $R_- > 1$ then $\lim_{t\to+\infty} \|x(t,.)\|_{L^1} = +\infty$ and $\lim_{t\to+\infty} y(t) = +\infty$ for every $(x_0, y_0) \in X_p$ (Figure 3).

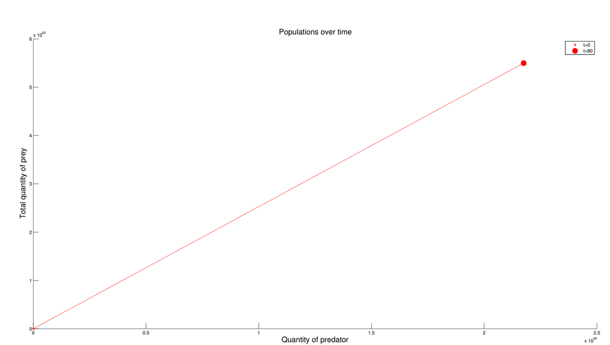

Figure 3: *Unbounded solutions*

When $R_0 > 1$ and $R_- < 1$, numerical simulations shows

- either convergence to a limit cycle (Figure 4);
- or convergence to E_2 (Figure 5).

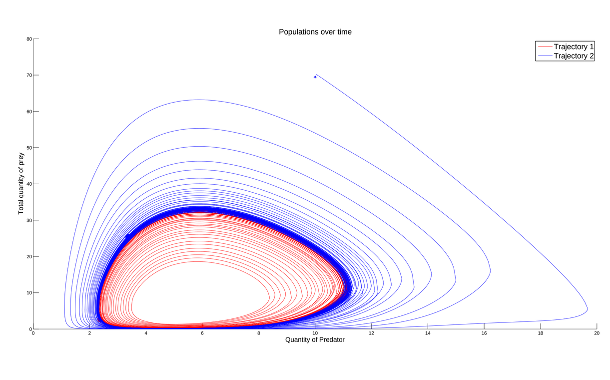

Figure 4: *Limit cycle*

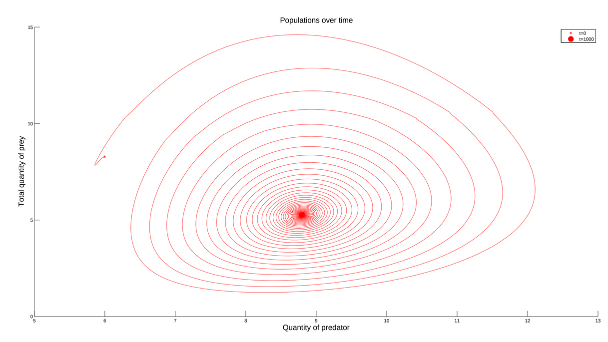

Figure 5: *Convergence to E_2*

$R_0 < 1$	$R_0 > 1$ and $R_- < 1$	$R_- > 1$
Extinction	Coexistence	Explosion

7. Perspectives

Let $Z = \mathcal{C}[-\tau, 0] \times \mathbb{R}$ and suppose that
$\mu \equiv \mu_0, \qquad \gamma(a) = \gamma_0 \mathbf{1}_{[\tau,+\infty[}(a), \qquad \beta(a) = \beta_0 \mathbf{1}_{[\tau,+\infty[}(a).$
We thus get the following delay system

$$\begin{cases} x'(t) = \beta_0 e^{-\mu_0\tau}x(t-\tau) - (\mu_0 + \gamma_0 y(t))x(t), \\ y'(t) = \alpha\gamma_0 x(t)y(t) - \delta y(t), \\ x(\theta) = \phi(\theta), \theta \in [-\tau, 0], \quad y(0) = y_0, \end{cases} \quad (2)$$

where $(\phi, y_0) \in Z$. When $R_0 > 1$, the equilibrium is

$$E^* = \left(\frac{\delta}{\alpha\gamma_0}, \frac{\beta_0 e^{-\mu_0\tau} - \mu_0}{\gamma_0}\right) = (x^*, y^*).$$

We get the characteristic equation $p(\lambda) + q(\lambda)e^{-\lambda\tau} = 0$,

$$\text{where } \begin{cases} p(\lambda) = \lambda^2 + \lambda\beta_0 e^{-\mu_0\tau} + \delta\gamma_0 y^*, \\ q(\lambda) = -\lambda\beta_0 e^{-\mu_0\tau}. \end{cases}$$

Absolute stability result from [1] implies

Theorem 5. (Local stability)
If $\tau\sqrt{\delta y^* \gamma_0}/2\pi \notin \mathbb{Z}$ then E^* is loc. asympt. stable.

Let $S = \{(\varphi, y) \in Z, \int_0^\tau \varphi > 0, y > 0\}$ then

Theorem 6. (Periodic solution)
For every $(\phi, y_0) \in S$ the solution of (2) converges to a τ-periodic function.

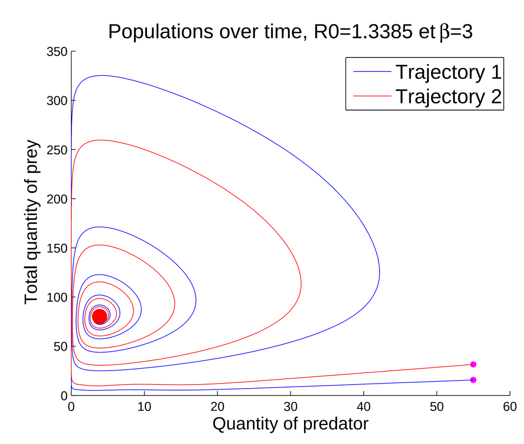

Figure 6: *Convergence to E^**

A future work will be to prove that the only τ-periodic solution is **constant**, hence the global stability of E_2 in S follows.

References

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